A nonmonotonic consequence relation for paradoxes of vagueness

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1 Introduction

- Paradoxes of vagueness are reasonings having intuitively true premises and false conlusion which follows classically from those premises.
- Paradoxes of vagueness are formulated in such a way that the conclusion of one is the classical negation of another like in the two formulations of the paradox of the heap below:

  (I) A heap is not formed with 1 grain.
  For all n: if a heap is not formed with n gains, then a heap is not formed with (n + 1) grains.
  ∴ A heap is not formed with 1.000.000 grains.

  (II) A heap is not formed with 1 grain.
  A heap is formed with 1.000.000 grains.
  ∴ There is a number n such that a heap is not formed with n grains and a heap is formed with (n + 1) grains.

- The only solution for those paradoxes is to prove that they are either invalid or unsound, that is to prove that conclusion does not follow from premises, or that a premise is not true.

2 Beckground

2.1 The Logic FOUR

- Two partial orderings $\leq_t$ and $\leq_k$ order the values: t, f, T , $\perp$: $f \leq_t T(\perp) \leq_t t$ and $\perp \leq_k t(f) \leq_k T$.
- Under both orderings we have a lattice: one with the meet denoted as $\wedge$ and join denoted as $\vee$; and the other with the meet denoted as $\otimes$ and the join $\oplus$.
- The operation of negation of FOUR is order reversing only w.r.t. the $\leq_t$ ordering.

- $f \otimes t = \perp$ and $f \oplus t = T$ (Fitting 1989).
- The set of desinated values $D = \{T, t\}$.
- A valuation which maps a given formula into an element of D is a model of the formula.
- $\Gamma \models_4 \alpha$ if every model of Γ is a model of α.
- $\models_4$ is monotonic but paraconsistent.

2.2 A Connective for Implication

$a \Rightarrow b = t$ if a is not a designated value;

$a \Rightarrow b = b$ otherwise (Arieli and Avron 1996).

- The function of valuation extends on the set of formulae with $\Rightarrow$ in the standard way.

2.3 Preferential Consequence Relation

- Preferential model: $(W, <)$, where W is a set of valuations on a given language (not necessarily all), and $<$ is an irreflexive and transitive relation over W (Makinson 2005).
- Preferential consequence relation: $\Gamma \models_< \alpha$ if $v(\alpha)$ is a designated value for every valuation $v \in W$ that is minimal among those in W that satisfy Γ, that is if every minimal preferential model of Γ is a model of α.
- $\Gamma \models_< \alpha$ is nonmonotonic.

3 The Main Result

3.1 Logical Forms of the Paradoxes

(I) $\neg H(g, 1)$
$[\neg H(g, 1) \Rightarrow \neg H(g, 2)] \wedge ... \wedge [\neg H(g, n) \Rightarrow \neg H(g, n{+}1)] \wedge ... \wedge [\neg H(g, 1.000.000 - 1) \Rightarrow \neg H(g, 1.000.000)]$
$\therefore \neg H(g, 1.000.000)$.

(II) $\neg H(g, 1)$
$H(g, 1.000.000)$
$\therefore [\neg H(g, 1) \wedge H(g, 2)] \vee ... \vee [\neg H(g, n) \wedge H(g, n{+}1)] \vee ... \vee [\neg H(g, 1.000.000 - 1) \wedge H(g, 1.000.000)]$.
$H(g, n)$ stands for *forms a heap with n grains*

3.2 Definition 1

$l_A(X) = t$ if A makes the statement X;

$l_A(X) = f$ if A makes the statemnt $\neg X$;

$l_A(X) = \perp$ otherwise.

and similarly for the speaker B.

3.3 Definition 2

The valuation v: $Fm \rightarrow (\{t, f, T, \perp\}, \neg, \wedge, \vee, \Rightarrow)$ is *intended* if it satisfies the following condition:

$$v(X) = l_A(X) \otimes l_B(X)$$

for each atomic sentence X.

3.4 Definition 3

A *preferential consequence relation* $\Gamma \Vdash_< \alpha$ holds iff every intended valuation with minimal number of non-classical assignments being a model of Γ is a model of α.

- Non-classical assignments: f: $Atom \rightarrow \{T, \perp\}$.
- If $v_1, v_2 \in W$, where W is the set of intended valuations, then

  $$v_1 < v_2$$

  if v_1 has less non-classical assignments than v_2.

- v is a most consistent intended model of Γ if $v \in W$ and there is no other model of Γ which has less non-classical assignments.
- $\Vdash_<$ is nonmonotonic and paraconsistent

3.5 Theorem

Theorem 1 *The Paradox (II) is invalid, while the Paradox (I) is unsound.*

3.6 Proof

The first premise of (II) has the value t , since

$$v[\neg H(g, 1)] = l_A[\neg H(g, 1)] \otimes l_B[\neg H(g, 1)]$$

$$\neg f \otimes \neg f = t \otimes t = t$$

The second premise of (II) has the value t, since

$$v[H(g, 1.000.000)] = l_A[H(g, 1.000.000)] \otimes l_B[H(g, 1.000.000)]$$

$$t \otimes t = t$$

But the conclusion of (II) does not have a designated value, since for at least one n:

$$v[H(g, n)] = l_A[H(g, n)] \otimes l_B[H(g, n)]$$

$$t \otimes f = \perp$$

$$v[H(g, n + 1)] = l_A[H(g, n + 1)] \otimes l_B[H(g, n + 1)]$$

$$t \otimes f = \perp$$

$$v[\neg H(g, n)] = \perp$$

and

$$v[\neg H(g,n) \wedge H(g,n+1)] = \bot$$

At last

$$sup\{f, \bot\} = \bot$$

since the conclusion is a disjunction with false disjuncts at its beginning and its end. This shows that there is a most consistent intended countermodel for the Paradox (II).
The major premise of the Paradox (I) has true conjuncts at the beginning and at the end of the conjunction. But for at least one n:

$$v[\neg H(g,n)] = t \otimes t = t$$

and

$$v[\neg H(g,n+1)] = t \otimes f = \bot$$

then

$$v[\neg H(g,n)] \Rightarrow \neg H(g,n+1)] = t \Rightarrow \bot = \bot$$

For at least one n:

$$v[\neg H(g,n)] = t \otimes f = \bot$$

and

$$v[\neg H(g,n+1)] = t \otimes f = \bot$$

then

$$v[\neg H(g,n)] \Rightarrow \neg H(g,n+1)] = \bot \Rightarrow \bot = t$$

For at least one n:

$$v[\neg H(g,n)] = t \otimes f = \bot$$

and

$$v[\neg H(g,n+1)] = f \otimes f = f$$

then

$$v[\neg H(g,n) \Rightarrow \neg H(g,n+1)] = \bot \Rightarrow f = t$$

At last

$$inf\{\bot, t\} = \bot$$

since the major premise of the Paradox (I) is a conjunction whose conjuncts have the value t or $\bot$. This shows that the Paradox (I) is unsound and ends the proof.

References

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